

Transition Amplitude within the Stochastic Quantization Scheme

– Perturbative Treatment –

Kazuya Yuasa* and Hiromichi Nakazato**

Department of Physics, Waseda University, Tokyo 169, Japan

Abstract

The quantum mechanical transition amplitudes are calculated perturbatively on the basis of the stochastic quantization method of Parisi and Wu. It is shown that the stochastic scheme reproduces the ordinary result for the amplitude and systematically incorporates higher-order effects, even at the lowest order.

* email address: 696L1136@mn.waseda.ac.jp

** email address: hiromici@mn.waseda.ac.jp

1 Introduction

Since its proposal, the stochastic quantization method of Parisi and Wu[1] has been widely applied to various fields in physics[2], quite straightforwardly at relatively early stages of its development and with some conceptual and/or technical innovation at later times. A scheme for obtaining the quantum mechanical transition amplitude on the basis of the stochastic quantization (SQ) was proposed by H. Hüffel and one of the present authors a few years ago[3] and is considered to belong to the latter type of applications: One may easily realize the difficulty in obtaining an *amplitude* from a *probability*, on which the stochastic process, even if fictitious in this scheme, is essentially based. In this paper, this scheme[3] is applied to a nonlinear system, for which no exact expression of the transition amplitude is known and we have to rely on a perturbative treatment. The purpose of the paper is to show how a perturbative treatment is possible within this scheme, to confirm its consistency with the ordinary one and then to find possible advantages over the latter.

After brief reviews of SQ in Sec. 2 and of the stochastic scheme for the transition amplitude in [3] in Sec. 3, we describe how to treat nonlinear systems within this scheme and propose a perturbative treatment in Sec. 4. Section 5 demonstrates some details of the calculation of the transition amplitudes for a quantum mechanical anharmonic oscillator. It will be clear that our treatment is equivalent to a perturbative expansion of the logarithm of the amplitude and therefore the amplitude thus obtained includes automatically a part of the higher-order contributions as an exponential form. The last section, Sec. 6, is devoted to a summary.

2 Review of the Stochastic Quantization Method

Let us briefly outline the ordinary prescription of the stochastic quantization method of Parisi and Wu[1, 2]. This quantization method is so designed as to give quantum mechanics as the thermal-equilibrium limit of a hypothetical stochastic process. For this purpose, the dynamical variable $x(t)$ ¹ is assumed to be a stochastic variable $x(t, s)$ with respect to a newly-introduced time, called the “fictitious time” s . Its dynamics is given by the Langevin equation²

$$\frac{\partial}{\partial s}x(t, s) = i\frac{\delta S}{\delta x(t, s)} + \eta(t, s), \quad (2.1)$$

where S is the classical action of the system and η is the Gaussian white noise, whose statistical properties are characterized by

$$\langle \eta(t, s) \rangle = 0, \quad \langle \eta(t, s)\eta(t', s') \rangle = 2\delta(t - t')\delta(s - s'), \quad etc. \quad (2.2)$$

We solve the Langevin equation (2.1) under some initial condition to get $x(t, s)$ as a functional of the noise η , calculate the equal-time correlation function $\langle x(t_1, s)x(t_2, s)\cdots \rangle$

¹For simplicity, we exclusively consider one-dimensional quantum mechanical systems here.

²We work in the Minkowski stochastic quantization scheme[4] throughout the paper.

by means of (2.2), and take the equilibrium limit $s \rightarrow \infty$. What we thereby obtain is shown to correspond to the ordinary vacuum expectation value $\langle 0 | T x(t_1) x(t_2) \cdots | 0 \rangle$.

This correspondence is most clearly seen in the Fokker-Planck picture. In this picture the stochastic average $\langle \cdots \rangle$ is expressed as a functional integral

$$\langle x(t_1, s) x(t_2, s) \cdots \rangle = \int \mathcal{D}x x(t_1) x(t_2) \cdots P[x; s] \quad (2.3)$$

with a (quasi-)probability distribution functional P which obeys the Fokker-Planck equation

$$\frac{\partial}{\partial s} P[x; s] = \int_{-\infty}^{\infty} dt \frac{\delta}{\delta x(t)} \left(\frac{\delta}{\delta x(t)} - i \frac{\delta S}{\delta x(t)} \right) P[x; s] \quad (2.4)$$

and is normalized as $\langle 1 \rangle = \int \mathcal{D}x P[x; s] = 1$. Clearly, the stationary solution of this equation is given by e^{iS} , which also serves as the equilibrium distribution if we adopt the so-called $i\varepsilon$ -prescription[4]. Under this prescription, the equal-time correlation function (2.3) approaches, as $s \rightarrow \infty$, the Feynman path integral, which is nothing but the vacuum expectation value in the canonical (operator) formalism:

$$\lim_{s \rightarrow \infty} \langle x(t_1, s) x(t_2, s) \cdots \rangle = \frac{\int \mathcal{D}x x(t_1) x(t_2) \cdots e^{iS}}{\int \mathcal{D}x e^{iS}} = \langle 0 | T x(t_1) x(t_2) \cdots | 0 \rangle. \quad (2.5)$$

3 The Stochastic Formula for Transition Amplitude

The transition amplitude is one of the fundamental quantities in quantum mechanics, for the amplitude plays a central role there. On the other hand, as one readily notices, the above stochastic scheme apparently provides us with those expectation values that are normalized in the sense of (2.5). This can be thought of as rooted in the essential property of the stochastic process, where the probability (density), not the amplitude, governs the dynamics. Therefore it does not seem trivial to derive amplitudes within the framework of the stochastic formalism of Parisi and Wu.

Let us first try to calculate the correlation function, under the boundary conditions

$$x(t_F, s) = x_F, \quad x(t_I, s) = x_I. \quad (3.1)$$

These conditions are considered to be one of the simplest extensions of those for the quantum mechanical transition amplitude $\langle x_F, t_F | x_I, t_I \rangle$. It would be natural to expect that the correlation function under the above boundary conditions $\langle x(t_1, s) x(t_2, s) \cdots \rangle_{bc}$ is expressed in the Fokker-Planck picture as

$$\langle x(t_1, s) x(t_2, s) \cdots \rangle_{bc} = \int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x x(t_1) x(t_2) \cdots P[x; s] \quad (3.2)$$

with the normalization condition $\langle 1 \rangle_{bc} = 1$. The quantity with the subscript bc is to be evaluated within SQ under the above boundary conditions (3.1). This normalization

condition implies that in the $s \rightarrow \infty$ limit, the correlation function approaches the following normalized quantity

$$\begin{aligned} \lim_{s \rightarrow \infty} \langle x(t_1, s) x(t_2, s) \cdots \rangle_{\text{bc}} &= \frac{\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x x(t_1) x(t_2) \cdots e^{iS}}{\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x e^{iS}} \\ &= \frac{\langle x_F, t_F | T x(t_1) x(t_2) \cdots | x_I, t_I \rangle}{\langle x_F, t_F | x_I, t_I \rangle}. \end{aligned} \quad (3.3)$$

It is such normalized expectation values (3.3) that we can directly calculate within this scheme. Remember that what we are interested in here is not the vacuum expectation value, but the transition amplitude $\langle x_F, t_F | x_I, t_I \rangle$ itself or, in other words, the normalization factor (*i.e.* the denominator in (3.3)) itself.

In [3], a formula for the transition amplitude within the framework of SQ is proposed and applied to several solvable cases. The formula is based on the following relations in quantum mechanics

$$-i \frac{\partial}{\partial t_I} \langle x_F, t_F | x_I, t_I \rangle = \langle x_F, t_F | H(t_I) | x_I, t_I \rangle \quad (3.4a)$$

and

$$i \frac{\partial}{\partial x_I} \langle x_F, t_F | x_I, t_I \rangle = \langle x_F, t_F | p(t_I) | x_I, t_I \rangle, \quad (3.4b)$$

where $H(t_I)$ and $p(t_I)$ are the Hamiltonian and the momentum operators, respectively. From (3.4a) we get

$$\frac{\partial}{\partial t_I} \ln \langle x_F, t_F | x_I, t_I \rangle = i \frac{\langle x_F, t_F | H(t_I) | x_I, t_I \rangle}{\langle x_F, t_F | x_I, t_I \rangle} \equiv i \langle H(t_I) \rangle_{\text{bc}}, \quad (3.5)$$

that has the solution

$$\langle x_F, t_F | x_I, t_I \rangle = c \exp \left[i \int^{t_I} dt_I \langle H(t_I) \rangle_{\text{bc}} \right], \quad (3.6a)$$

with a factor c dependent on x_F , x_I and t_F . Along the same line of thought, we obtain from (3.4b)

$$\langle x_F, t_F | x_I, t_I \rangle = \tilde{c} \exp \left[-i \int^{x_I} dx_I \langle p(t_I) \rangle_{\text{bc}} \right] \quad (3.6b)$$

with a factor \tilde{c} dependent on x_F , t_F and t_I . The transition amplitude $\langle x_F, t_F | x_I, t_I \rangle$ has thus been related to the (normalized) expectation value of the Hamiltonian $\langle H(t_I) \rangle_{\text{bc}}$ and to that of the momentum $\langle p(t_I) \rangle_{\text{bc}}$, which are obtainable as the equilibrium limits of $\langle H(t_I, s) \rangle_{\text{bc}}$ and $\langle p(t_I, s) \rangle_{\text{bc}}$ in SQ.³

³Equation (3.6a) is the formula presented in [3]. On the other hand, though it is not explicitly presented there, (3.6b) is surely used in the practical calculations.

Notice that these relations are nothing but two of the expressions of the ordinary variational principle and there are two more formulae, *i.e.*, the formulae relating the transition amplitude to $\langle H(t_F) \rangle_{bc}$ and $\langle p(t_F) \rangle_{bc}$. For most systems, however, the above two formulae (3.6) are sufficient to determine c and \tilde{c} except for a constant factor independent of t_F , t_I , x_F and x_I . In fact, we can determine the x_I -dependence of c from (3.6b) and the t_I -dependence of \tilde{c} from (3.6a). Furthermore, their x_F - and t_F -dependences are fixed from the fact that $\langle x_F, t_F | x_I, t_I \rangle$ is a function of $T \equiv t_F - t_I$ (translational invariance) and is a symmetric function of x_F and x_I (time-reversal invariance). The constants c and \tilde{c} are thus completely determined if we fix the remaining factor by requiring $\langle x_F, t_F | x_I, t_I \rangle$ to approach a Dirac $\delta(x_F - x_I)$ as $T \rightarrow 0$.

4 A Perturbative Treatment

The above formulae (3.6) have been applied in [3] to linear systems to derive transition amplitudes within the framework of SQ. It is demonstrated that the stochastic scheme can reproduce the correct results for the amplitudes. In the following, we shall develop a perturbative method in order to treat nonlinear systems within this scheme.

Since all that is needed in deriving the amplitude in SQ are the expectation values of H and p , according to (3.6a) and (3.6b), a natural strategy would be to evaluate these quantities as a power series in the coupling constant. For systems in which the Hamiltonian H is composed of a kinetic part $p^2/2M$ and a potential part $V(x)$, what essentially remains to be evaluated is the expectation value of the former, owing to the boundary conditions (3.1)

$$\langle H(t_I) \rangle_{bc} = \frac{1}{2M} \langle p^2(t_I) \rangle_{bc} + V(x_I). \quad (4.1)$$

We have to perturbatively evaluate the expectation values of momentum and squared momentum. Though this can be done within the framework of the phase space formalism of SQ[5, 3], we prefer to work in ordinary configuration space for simplicity. It is not difficult to derive the following relations

$$\langle p(t_I) \rangle_{bc} = M \lim_{t \rightarrow t_I} \partial_t \lim_{s \rightarrow \infty} \langle x(t, s) \rangle_{bc} \quad (4.2a)$$

and

$$\langle p^2(t_I) \rangle_{bc} = M^2 \lim_{t_1, t_2 \rightarrow t_I} \partial_{t_1} \partial_{t_2} \lim_{s \rightarrow \infty} \langle x(t_1, s) x(t_2, s) \rangle_{bc}, \quad (4.2b)$$

on the basis of the phase-space Langevin equations.⁴ Thus the problem is reduced to the evaluation of the correlation functions, $\langle x(t, s) \rangle_{bc}$ and $\langle x(t_1, s) x(t_2, s) \rangle_{bc}$: We solve the Langevin equation (2.1) perturbatively and get its solution $x(t, s)$ as a power series in the coupling constant. As in the solvable cases treated in [3], the boundary conditions

⁴The second relation just corresponds to Feynman's time-splitting procedure in evaluating the average square velocity[6].

(3.1) shall be taken into account by the classical solution and the remaining part (to be quantized in terms of the Langevin equation) is subject to the “zero boundary conditions” and is expressed as a Fourier series.

It is worth mentioning here that the perturbative treatment on the basis of the formulae (3.6a) and (3.6b) differs from the ordinary one: What is really evaluated perturbatively is not the amplitude itself, as in the ordinary case, but its logarithm. Therefore we can expect that even the lowest-order value of $\langle H(t_I) \rangle_{bc}$ or $\langle p(t_I) \rangle_{bc}$ will systematically yield higher-order contributions to the amplitude.

5 An Example: Anharmonic Oscillator

Let us consider an anharmonic oscillator described by the action

$$S = \int_{t_I}^{t_F} dt \left(\frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \omega^2 x^2 - \frac{g}{4} x^4 \right) \quad (5.1)$$

and calculate the transition amplitude $\langle x_F, t_F | x_I, t_I \rangle$ according to the above stochastic scheme. The Langevin equation (2.1) reads

$$\partial_s x(t, s) = -iM(\partial_t^2 + \omega^2)x(t, s) - igx^3(t, s) + \eta(t, s) \quad (5.2)$$

and we write its solution as the sum of the solution for a harmonic oscillator ($g = 0$) $x^{(0)}$ and a remaining part

$$x(t, s) = x^{(0)}(t, s) + x'(t, s). \quad (5.3)$$

The free solution $x^{(0)}$ is composed of two terms

$$x^{(0)}(t, s) = x_{cl}^{(0)}(t) + x_Q^{(0)}(t, s), \quad (5.4)$$

where $x_{cl}^{(0)}$ is the solution of the classical equation of motion

$$(\partial_t^2 + \omega^2)x_{cl}^{(0)}(t) = 0 \quad \text{with} \quad x_{cl}^{(0)}(t_F) = x_F, \quad x_{cl}^{(0)}(t_I) = x_I \quad (5.5)$$

and $x_Q^{(0)}$ that of a free Langevin equation

$$\partial_s x_Q^{(0)}(t, s) = -iM(\partial_t^2 + \omega^2)x_Q^{(0)}(t, s) + \eta(t, s) \quad \text{with} \quad x_Q^{(0)}(t_F) = x_Q^{(0)}(t_I) = 0, \quad (5.6)$$

where the last conditions are called “zero boundary conditions.” They are readily calculated to be

$$x_{cl}^{(0)}(t) = \frac{1}{\sin \omega T} [x_F \sin \omega(t - t_I) + x_I \sin \omega(t_F - t)] \quad (5.7)$$

and

$$x_Q^{(0)}(t, s) = \int_{t_I}^{t_F} dt' \int_{-\infty}^{\infty} ds' G_{bc}^{(0)}(t, t'; s - s') \eta(t', s'), \quad (5.8)$$

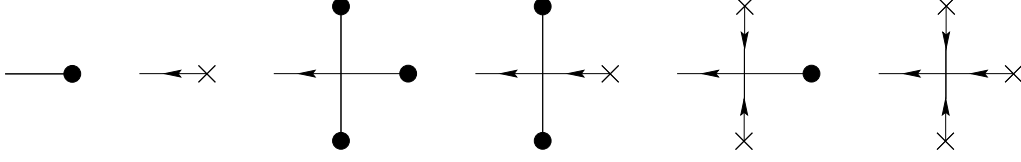


Fig. 1. Graphical representation of the perturbation series of x .

where

$$G_{\text{bc}}^{(0)}(t, t'; s - s') = \theta(s - s') \frac{2}{T} \sum_{n=1}^{\infty} \sin \frac{n\pi}{T}(t - t_1) \sin \frac{n\pi}{T}(t' - t_1) e^{-iM\omega^2 \left[1 - \left(\frac{n\pi}{\omega T}\right)^2\right](s - s')} \quad (5.9)$$

is the free Green function which respects the “zero boundary conditions” (5.6). The retarded Green function $G_{\text{bc}}^{(0)}$ satisfies

$$\left[\partial_s + iM(\partial_t^2 + \omega^2)\right] G_{\text{bc}}^{(0)}(t, t'; s - s') = \delta(t - t')\delta(s - s') \quad (5.10a)$$

and the boundary conditions

$$G_{\text{bc}}^{(0)}(t, t'; s - s') = 0 \quad \text{for } t \text{ (or } t') = t_{\text{F}}, t_{\text{I}} \text{ or } s < s'. \quad (5.10b)$$

Here we have set the initial fictitious time at $s_0 = -\infty$, in order to achieve the long-time limit automatically. This choice makes any initial-value dependence of the solution $x_{\text{Q}}^{(0)}$ irrelevant, for the system is considered to be in equilibrium already at finite s . Now we solve the Langevin equation (5.2) to get a recursive relation for $x(t, s)$

$$x(t, s) = x_{\text{cl}}^{(0)}(t) + x_{\text{Q}}^{(0)}(t, s) - ig \int_{t_{\text{I}}}^{t_{\text{F}}} dt' \int_{-\infty}^{\infty} ds' G_{\text{bc}}^{(0)}(t, t'; s - s') x^3(t', s'), \quad (5.11)$$

from which we obtain the perturbation series

$$x = x_{\text{cl}}^{(0)} + x_{\text{Q}}^{(0)} - ig \int_{t_{\text{I}}}^{t_{\text{F}}} dt' \int_{-\infty}^{\infty} ds' G_{\text{bc}}^{(0)} \left(x_{\text{cl}}^{(0)3} + 3x_{\text{cl}}^{(0)2} x_{\text{Q}}^{(0)} + 3x_{\text{cl}}^{(0)} x_{\text{Q}}^{(0)2} + x_{\text{Q}}^{(0)3} \right) + O(g^2). \quad (5.12)$$

We represent the series (5.12) diagrammatically as in Fig. 1, where $x_{\text{cl}}^{(0)}$, $G_{\text{bc}}^{(0)}$, and η are denoted by a line with a dot, a line with an arrow, and a cross, respectively, and at each vertex, a factor $-ig$ and integration variables t' and s' are attached. The stochastic average over η (2.2) combines all the crosses in pairs in all possible ways. We also have to integrate over the vertex times t' and s' (see (5.12)).

According to the formula (4.2a), we need to evaluate $\langle x \rangle_{\text{bc}}$ in order to get $\langle p(t_{\text{I}}) \rangle_{\text{bc}}$, to which the three diagrams in Fig. 2 contribute up to $O(g)$, where a simple line without

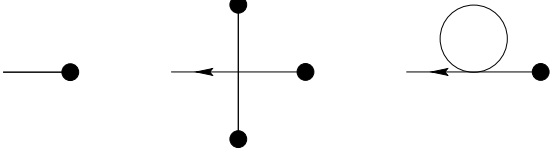


Fig. 2. Stochastic diagrams contributing to $\langle x \rangle_{\text{bc}}$ up to $O(g)$.

any arrow or dot denotes the free two-point correlation function

$$\begin{aligned}
D_{\text{bc}}^{(0)}(t, t'; s - s') &\equiv \langle x_{\text{Q}}^{(0)}(t, s) x_{\text{Q}}^{(0)}(t', s') \rangle_{\text{bc}} \\
&= 2 \int_{t_{\text{I}}}^{t_{\text{F}}} dt_1 \int_{-\infty}^{\infty} ds_1 G_{\text{bc}}^{(0)}(t, t_1; s - s_1) G_{\text{bc}}^{(0)}(t', t_1; s' - s_1) \\
&= \frac{2i}{M\omega^2 T} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{n\pi}{\omega T}\right)^2 - 1} \sin \frac{n\pi}{T}(t - t_{\text{I}}) \sin \frac{n\pi}{T}(t' - t_{\text{I}}) e^{-iM\omega^2 \left[1 - \left(\frac{n\pi}{\omega T}\right)^2\right] |s - s'|}.
\end{aligned} \tag{5.13}$$

These three diagrams correspond to the following terms in $\langle x \rangle_{\text{bc}}$

$$\begin{aligned}
\langle x(t, s) \rangle_{\text{bc}} &= x_{\text{cl}}^{(0)}(t) - ig \int_{t_{\text{I}}}^{t_{\text{F}}} dt' \int_{-\infty}^{\infty} ds' G_{\text{bc}}^{(0)}(t, t'; s - s') \left(x_{\text{cl}}^{(0)3}(t') + 3x_{\text{cl}}^{(0)}(t') D_{\text{bc}}^{(0)}(t', t'; 0) \right) \\
&\quad + O(g^2).
\end{aligned} \tag{5.14}$$

A relation between $G_{\text{bc}}^{(0)}$ and $D_{\text{bc}}^{(0)}$ (see the Appendix) enables us to perform the integration over s' and yields

$$\langle x(t) \rangle_{\text{bc}} = x_{\text{cl}}^{(0)}(t) - ig \int_{t_{\text{I}}}^{t_{\text{F}}} dt' \Delta_{\text{bc}}(t, t') \left(x_{\text{cl}}^{(0)3}(t') + 3x_{\text{cl}}^{(0)}(t') \Delta_{\text{bc}}(t', t') \right) + O(g^2), \tag{5.15}$$

where $\Delta_{\text{bc}}(t, t') \equiv D_{\text{bc}}^{(0)}(t, t'; 0)$ and its explicit form is[3]

$$\Delta_{\text{bc}}(t, t') = \frac{i}{M\omega \sin \omega T} \sin \omega(t_{\text{F}} - \max(t, t')) \sin \omega(\min(t, t') - t_{\text{I}}). \tag{5.16}$$

On the other hand, the expectation value $\langle p^2(t_{\text{I}}) \rangle_{\text{bc}}$ is calculated from the correlation function $\langle xx \rangle_{\text{bc}}$ through the relation (4.2b). At the lowest order, there are three types of connected diagrams, shown in Fig. 3, contributing to $\langle xx \rangle_{\text{bc}}$ and we have, after the integration over s' ,

$$\langle x(t)x(t') \rangle_{\text{bc}} = \langle x(t) \rangle_{\text{bc}} \langle x(t') \rangle_{\text{bc}} + \Delta_{\text{bc}}(t, t')$$

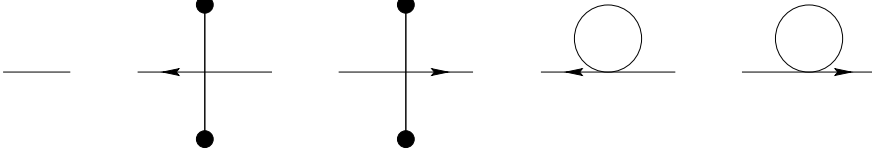


Fig. 3. Connected stochastic diagrams contributing to $\langle xx \rangle_{bc}$ up to $O(g)$.

$$\begin{aligned}
& - 3ig \int_{t_I}^{t_F} dt_1 \Delta_{bc}(t, t_1) \left(x_{cl}^{(0)2}(t_1) + \Delta_{bc}(t_1, t_1) \right) \Delta_{bc}(t_1, t') \\
& + O(g^2).
\end{aligned} \tag{5.17}$$

We are now in a position to write the explicit form of the transition amplitude $\langle x_F, t_F | x_I, t_I \rangle$ for the present system. Since the Hamiltonian of this system is given by

$$H = \frac{1}{2M} p^2 + \frac{1}{2} M \omega^2 x^2 + \frac{g}{4} x^4, \tag{5.18}$$

we have

$$\langle H(t_I) \rangle_{bc} = \frac{1}{2M} \langle p^2(t_I) \rangle_{bc} + \frac{1}{2} M \omega^2 x_I^2 + \frac{g}{4} x_I^4, \tag{5.19}$$

where the last two terms are a consequence of the boundary conditions (3.1) for x . From (5.15) and (5.17) we calculate $\langle p(t_I) \rangle_{bc}$ and $\langle H(t_I) \rangle_{bc}$ through (5.19) and (4.2), and insert them into the formulae (3.6). In this way, we obtain $\langle x_F, t_F | x_I, t_I \rangle$ in the exponential form

$$\langle x_F, t_F | x_I, t_I \rangle = \sqrt{\frac{M\omega}{2i\pi \sin \omega T}} e^{iS_0 + iS_1 + O(g^2)}, \tag{5.20a}$$

where

$$S_0 = \frac{M\omega}{2 \sin \omega T} \left[(x_F^2 + x_I^2) \cos \omega T - 2x_F x_I \right] \tag{5.20b}$$

and

$$\begin{aligned}
S_1 = & g \frac{3}{32M^2\omega^3 \sin^2 \omega T} \left(3\omega T - 3 \sin \omega T \cos \omega T - 2\omega T \sin^2 \omega T \right) \\
& + ig \frac{3}{16M\omega^2 \sin^3 \omega T} \left[(x_F^2 + x_I^2) \left(3\omega T \cos \omega T - 3 \sin \omega T + \sin^3 \omega T \right) \right. \\
& \quad \left. - 2x_F x_I \left(3\omega T - 3 \sin \omega T \cos \omega T - 2\omega T \sin^2 \omega T \right) \right] \\
& - g \frac{1}{32\omega \sin^4 \omega T} \left[(x_F^4 + x_I^4) \left(3\omega T - 3 \sin \omega T \cos \omega T - 2 \sin^3 \omega T \cos \omega T \right) \right. \\
& \quad - 4(x_F^3 x_I + x_F x_I^3) \left(3\omega T \cos \omega T - 3 \sin \omega T + \sin^3 \omega T \right) \\
& \quad \left. + 6x_F^2 x_I^2 \left(3\omega T - 3 \sin \omega T \cos \omega T - 2\omega T \sin^2 \omega T \right) \right].
\end{aligned} \tag{5.20c}$$

Notice again that the constant c in (3.6a) contains enough information to fix the $T = t_F - t_I$ dependence of \tilde{c} in (3.6b). On the other hand, the latter can fix the $x_F(x_I)$ dependence of the former. The remaining factor has been fixed from the normalization condition at $t_F = t_I$. The result (5.20) coincides with the ordinary perturbative expression for the transition amplitude if it is expanded as a power series in the coupling constant g , as it should be.

6 Summary

In this paper, we have proposed a perturbative treatment of transition amplitudes within the stochastic quantization scheme and have applied it to an anharmonic oscillator. Our treatment is based on the Langevin equation: We solve it iteratively to get its solution x perturbatively under given boundary conditions. This enables us to calculate $\langle H(t_I) \rangle_{bc}$ and $\langle p(t_I) \rangle_{bc}$ as power series in g , from which the transition amplitude is obtained by making use of the formulae (3.6). Even though up to the order of interest ($O(g)$, in this case), they coincide with each other, the expression (5.20) is different from the one usually obtained, for example, in the path-integral formulation: In this scheme, it is the argument of an exponential function that is expanded as a power series in g . In this sense, we can say that the present stochastic scheme systematically incorporates higher-order effects.

It is worth stressing again that the quantity to be calculated perturbatively in this scheme is not the amplitude itself, but its logarithm. This is rooted in the exponential form of the formulae (3.6). These formulae are quite remarkable, in the sense that all quantum fluctuations are put together in one exponent ($\int \langle H(t_I) \rangle_{bc} dt_I$ or $\int \langle p(t_I) \rangle_{bc} dx_I$), which is contrasted to the usual situation in the path-integral quantization where fluctuations manifest themselves as different paths. Though it is not easy to get an intuitive physical image (if any) of these exponents, it might be helpful to remind that a similar relation exists between the ordinary generating functional and its counterpart for connected diagrams. The study, along such a line of thought, might lead us to a deeper understanding of the stochastic scheme itself and shed new light on the meaning of quantization.

Acknowledgements

The authors acknowledge useful and helpful discussions with Prof. I. Ohba and Dr. Y. Yamanaka. This work is partly supported by the Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, Japan (#07804017).

Appendix

We shall prove the relation

$$\int_{-\infty}^{\infty} ds' G_{\text{bc}}^{(0)}(t, t'; s - s') = \Delta_{\text{bc}}(t, t'), \quad (\text{A.1})$$

by means of which we get (5.15). Here Δ_{bc} is defined as

$$\Delta_{\text{bc}}(t, t') \equiv D_{\text{bc}}^{(0)}(t, t'; s - s') \Big|_{s=s'}, \quad (\text{A.2})$$

with

$$D_{\text{bc}}^{(0)}(t, t'; s - s') \equiv \langle x_{\text{Q}}^{(0)}(t, s) x_{\text{Q}}^{(0)}(t', s') \rangle_{\text{bc}}. \quad (\text{A.3})$$

This function has the following explicit form (see (2.2), (5.8) and (5.9))

$$\begin{aligned} D_{\text{bc}}^{(0)}(t, t'; s - s') &= 2 \int_{t_1}^{t_F} dt_1 \int_{-\infty}^{\infty} ds_1 G_{\text{bc}}^{(0)}(t, t_1; s - s_1) G_{\text{bc}}^{(0)}(t', t_1; s' - s_1) \\ &= \frac{2i}{M\omega^2 T} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{n\pi}{\omega T}\right)^2 - 1} \sin \frac{n\pi}{T}(t - t_1) \sin \frac{n\pi}{T}(t' - t_1) e^{-iM\omega^2 \left[1 - \left(\frac{n\pi}{\omega T}\right)^2\right] |s - s'|}. \end{aligned} \quad (\text{A.4})$$

It is easy to recognize the existence of a relation between $G_{\text{bc}}^{(0)}$ in (5.9) and $D_{\text{bc}}^{(0)}$ in (A.4)

$$G_{\text{bc}}^{(0)}(t, t'; s - s') = \theta(s - s') \frac{\partial}{\partial s'} D_{\text{bc}}^{(0)}(t, t'; s - s'). \quad (\text{A.5})$$

Hence the integration in (A.1) becomes trivial. The above relation is nothing but the “golden rule,” as first observed in [7] and later interpreted as a realization of the fluctuation dissipation theorem[8].

References

- [1] G. Parisi and Y.-S. Wu, Sci. Sin. **24** (1981), 483.
- [2] For reviews, see
P. Damgaard and H. Hüffel, Phys. Rep. **152** (1987), 227 and
M. Namiki, *Stochastic Quantization* (Springer, 1992).
- [3] H. Hüffel and H. Nakazato, Mod. Phys. Lett. **A9** (1994), 2953.
- [4] H. Hüffel and H. Rumpf, Phys. Lett. **B148** (1984), 104.
E. Gozzi, Phys. Lett. **B150** (1985), 119.
H. Nakazato and Y. Yamanaka, Phys. Rev. **D34** (1986), 492.
H. Nakazato, Prog. Theor. Phys. **77** (1987), 20.

- [5] I. Ohba, Prog. Theor. Phys. **77** (1987), 1267.
- [6] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw Hill, 1965).
- [7] H. Nakazato, M. Namiki, I. Ohba and K. Okano, Prog. Theor. Phys. **70** (1983), 298.
- [8] S. Chaturvedi, A. K. Kapoor and V. Srinivasan, Z. Phys. **B57** (1984), 249.
E. Gozzi, Phys. Rev. **D30** (1984), 1218.
L. Garrido and M. San Miguel, Prog. Theor. Phys. **59** (1978), 40; **59** (1978), 55.
H. Nakazato, Phys. Rev. **D48** (1993), 5838.